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A closer look at two AdS_4 branes in an AdS_5 bulk

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Abstract: We investigate a scenario with two AdS_4 branes in an AdS_5 bulk. In this scenario there are two gravitons and we investigate the role played by each of them for different positions of the second brane. We show that both gravitons play a significant role only when the turn-around point in the warp factor is approximately equidistant from both branes. We find that the ultralight mode becomes heavy as the second brane approaches the turn-around point, and the physics begins to resemble that of the RS model. Thus we demonstrate the crucial role played by the turn-around in the warp factor in enabling the presence of both gravitons.

Keywords: Field Theories in Higher Dimensions, Extra Large Dimensions, p-branes.

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Contents

1. Introduction

There has been much interest recently in brane world scenarios and their implications for large extra dimensions. Whilst the SM fields are confined to the brane, gravity propogates in the 5D bulk. In conventional models, the Newton potential between two masses on the brane results from the exchange of a massless graviton and a tower of massive KK modes. However, the model proposed by Karch and Randall [1] gives the first instance where 4D gravity is mediated by a massive (though light) graviton, and there is no massless mode. In this scenario the brane has a small negative 4D cosmological constant on it. The warp factor blows up far from the brane, and the geometry includes the boundary of AdS_5 and thus the massless mode is nonnormalizable. However the first KK mode is ultralight in comparison to the rest of the tower, and it is this mode which now plays the role of the graviton.

The introduction of a second brane to cut off the extra dimension means the normalization of the massless mode is made finite. We are then left with a situation where we have two gravitons — a massless mode and an ultralight mode (we are assuming a small 4D cosmological constant). This model was investigated by Kogan et al [2] in the case that the two branes are equidistant from the turn-around point in the warp factor. The concept of having two gravitons with interchanging roles is fascinating, and merits further investigation, which is the subject of this paper.

We investigate what happens to the two gravitons when the second brane is placed at an arbitrary postion, then we look at three specific limits in order to simplify matters. We look at whether the massless and ultralight modes couple with equal strength to the branes, and where they are concentrated in these limits. We are particularly interested in what happens when the second brane approaches the turn-around point in the warp factor. In this case we find the mass of the ultralight mode blows up and we are left with only the massless graviton. This presents a scenario similar to the RS model, demonstrating the necessity of the kink in the warp factor to producing a situation involving two gravitons (or more specifically an ultralight mode). Conversely when the second brane is taken to infinity, the massless mode is no longer normalizable and we are left with only the ultralight mode as in [1]. We also calculate the Newton potential. We investigate the origins of having two gravitons in the theory by seeing what happens when we bring two AdS_4 branes from infinity to a finite separation. Finally we look at what happens to the two gravtions in the Minkowski limit, and how they combine to form the massless mode of the Randall-Sundrum model.

2. The background

We have 2 branes at positions $y = 0, y_c$. The action for this situation is:

$$S = \int d^4x \int_{-y_c}^{y_c} dy\sqrt{-g}(-\Lambda + 2M^3 R) + \sum_i \int_{y=L_i} d^4x V_i \sqrt{\hat{g}^{(i)}} \tag{2.1}$$

where $\hat{g}^{(i)}$ is the induced metric on the branes; Λ, M are the 5-d cosmological constant and fundamental gravity scale respectively.

The metric is as follows:

$$\begin{aligned} ds^2 &= e^{2A(y)} g_{ij} dx^i dx^j + dy^2 \\ e^{A(y)} &= \frac{\cosh(\frac{y_0 - |y|}{L})}{\cosh(\frac{y_0}{L})}\,. \end{aligned} \tag{2.2}$$

where y_0 is the turn-around point in the warp factor and $1/L = \sqrt{-\Lambda/24M^3}$. The warp factor has been normalized to one at the position of the first brane (at 0). The induced metric g_{ij} is described by

$$g_{ij} dx^i dx^j = \frac{1}{-\bar{\Lambda}_4} \frac{1}{\cos^2\rho}(dt^2 - d\rho^2 - \sin^2\rho \, d\Omega_2^2) \tag{2.3}$$

and the radius of AdS_4, l_4 is given by

$$-\bar{\Lambda}_4 \sim \frac{1}{l_4^2}$$
$$l_4 = L\cosh\left(\frac{y_0}{L}\right) \tag{2.4}$$

where $\bar{\Lambda}_4$ is the 4d cosmological constant on the first brane.

The brane tensions depend on the derivative of the warp factor at the position of each brane. They are:

$$V_1 = \frac{3}{L}\tanh\left(\frac{y_0}{L}\right)$$
$$V_2 = \frac{3}{L}\tanh\left(\frac{y_c - y_0}{L}\right). \tag{2.5}$$

We can find the 4-d cosmolgical constant on each brane i in terms of the tension:

$$-\Lambda_4^i \sim \frac{1}{L^2}\frac{1}{\cosh^2(\frac{y_i - y_0}{L})}$$
$$= \frac{1}{L^2}\left(1 - \left(\frac{L}{3}\right)^2 V_i^2\right). \tag{2.6}$$

We wish for the 4-d cosmological constant on our brane to be small, as this is essential if we want the first KK mode to be ultralight. If we live on the brane at $y = 0$ this corresponds to large y_0/L, which we assume from now on. Then the brane tension $V_1 \to 3/L$.

We will be using the conformal coordinate z which is related to y by

$$\tan\left(\frac{1}{2\hat{L}}(|z| - z_0)\right) \equiv \tanh\left(\frac{1}{2L}(|y| - y_0)\right)$$
$$\frac{1}{\hat{L}} = \frac{1}{L\cosh(\frac{y_0}{L})} = \frac{\cos(\frac{z_0}{\hat{L}})}{L}. \tag{2.7}$$

3. The 2 gravitons

3.1 The massless mode

We start by looking at the role of the massless mode. Its wavefunction is

$$\hat{\psi}^{(0)}(z) = \frac{A}{\cos^{\frac{3}{2}}(\frac{(|z| - z_0)}{\hat{L}})} \tag{3.1}$$

where the normalization constant A can be obtained from: $\int_{-z_c}^{z_c} dz\hat{\psi}^2 = 1$ so

$$A^2 = \frac{1}{\hat{L}\left(\frac{\sin((z_c - z_0)/\hat{L})}{\cos^2((z_c - z_0)/\hat{L})} + \frac{\sin(z_0/\hat{L})}{\cos^2(z_0/\hat{L})} + \ln\tan\left(\frac{1}{2\hat{L}}(z_c - z_0) + \frac{\pi}{4}\right) + \ln\tan\left(\frac{1}{2\hat{L}}(z_0) + \frac{\pi}{4}\right)\right)}. \tag{3.2}$$

Let us consider three specific values for the position of the second brane, which are sufficient to give us an idea of what is happening in general.

We investigate what happens when the second brane is sent to infinity, when it is at $2z_0$ (symmetric configuration) and in the limit that it approaches the turn-around in the warp factor, z_0.

3.1.1 $z_c = z_\infty$

First we send the 2nd brane to infinity, that is to a position z_∞ in conformal coordinates, given by:

$$\frac{(z_\infty - z_0)}{\hat{L}} = \frac{\pi}{2}$$

then

$$A^2 \sim \frac{1}{\hat{L}\left(\frac{1}{\cos^2(\frac{\pi}{2})} + \ln\tan(\frac{\pi}{2})\right)} \to 0\,. \tag{3.3}$$

We see the massless mode is not normalizable, it does not exist.

What happens to its wavefunction?

$$\hat{\psi}^{(0)}(z)^2 = \frac{A^2}{\cos^3((|z| - z_0)/\hat{L})} \to 0 \tag{3.4}$$

for all z except at $z_c = z_\infty$ where we get

$$\frac{1}{\cos^3(\frac{\pi}{2})} \frac{1}{\frac{1}{\cos^2(\frac{\pi}{2})} + \cdots} \to \infty \tag{3.5}$$

thus the mode is concentrated at the 2nd brane at infinity.

3.1.2 $z_c = 2z_0$

What happens if the branes are placed in a symmetric configuration around the turn-around in the warpfactor, as in [2] ($z_c = 2z_0$)? Then

$$\begin{aligned} \hat{\psi}^{(0)}(z)^2 &= \frac{\cos(z_0/\hat{L})}{2L\cos^3((|z| - z_0)/\hat{L})} \frac{1}{\frac{\sin(z_0/\hat{L})}{\cos^2(z_0/\hat{L})} + \ln\tan(\frac{z_0}{2\hat{L}} + \frac{\pi}{4})} \\ \cos\left(\frac{z_0}{\hat{L}}\right) &\equiv \frac{1}{\cosh(y_0/L)} \approx 2e^{-y_0/L} \end{aligned} \tag{3.6}$$

for large y_0/L, so:

$$\hat{\psi}^{(0)}(z)^2 \approx \frac{1}{2L} \frac{\cos^3(z_0/\hat{L})}{\cos^3((|z| - z_0)/\hat{L})} \tag{3.7}$$

thus

$$\hat{\psi}^{(0)}(z_c = 2z_0) = \hat{\psi}^{(0)}(0) = \frac{1}{2L}\,. \tag{3.8}$$

The massless mode is equally distributed on both branes as expected.

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3.1.3 $z_c = z_0$

What happens if the 2nd brane actually approaches the turn-around in the warpfactor?

$$\begin{aligned}\hat{\psi}^{(0)}(z)^2 &= \frac{\cos(z_0/\hat{L})}{L\cos^3((|z|-z_0)/\hat{L})}\frac{1}{\frac{\sin(z_0/\hat{L})}{\cos^2(z_0/\hat{L})}+\ln\tan(\frac{z_0}{2\hat{L}}+\frac{\pi}{4})} \\ &\approx \frac{\cos^3(z_0/\hat{L})}{L\cos^3((|z|-z_0)/\hat{L})} \end{aligned} \tag{3.9}$$

so $\hat{\psi}^{(0)}(0) = 1/L$, $\hat{\psi}^{(0)}(z_c = z_0) \sim \frac{8}{L}\,e^{-\frac{3}{L}y_0}$ which is very small. The massless mode is now concentrated on the first (Planck) brane. For comparison, in RS1 the massless mode is also concentrated on the Planck brane.

3.2 The ultralight 1st KK mode

Now that we have studied the massless mode in some detail, let us look at the ultralight 1st KK mode, which plays the role of the 2nd graviton in this scenario.

The KK states obey:

$$\begin{aligned}[\partial_z^2 - V(z)]\hat{\psi}^{(n)}(z) &= -m^2\hat{\psi}^{(n)}(z) \\ V(z) &= -\frac{9}{4\hat{L}^2}+\frac{15}{4\hat{L}^2}\frac{1}{\cos^2((|z|-z_0)/\hat{L})} - \\ &\quad -\frac{3}{\hat{L}}\left[\tan\left(\frac{z_0}{\hat{L}}\right)\delta(z)+\tan\left(\frac{(z_c-z_0)}{\hat{L}}\right)\delta(z-z_c)\right] \end{aligned} \tag{3.10}$$

which can be shown to have the form of a hypergeometric differential equation which has the following solution:

$$\begin{aligned}\hat{\psi}^{(n)} &= \cos^{\frac{5}{2}}\left(\frac{(|z|-z_0)}{\hat{L}}\right)\left[C_1 F_{\frac{5}{4}} - C_2\sin\left(\frac{(|z|-z_0)}{\hat{L}}\right)F_{\frac{7}{4}}\right] \\ F_{\frac{5}{4}} &= F\left(\frac{5}{4}+\frac{1}{2}\sqrt{\left(\frac{m}{\hat{k}}\right)^2+\frac{9}{4}}, \frac{5}{4}-\frac{1}{2}\sqrt{\left(\frac{m}{\hat{k}}\right)^2+\frac{9}{4}}, \frac{1}{2}, \sin^2\left(\frac{(|z|-z_0)}{\hat{L}}\right)\right) \\ &= F\left(\alpha,\beta,\gamma,\sin^2\left(\frac{(|z|-z_0)}{\hat{L}}\right)\right) \\ F_{\frac{7}{4}} &= F\left(\frac{7}{4}+\frac{1}{2}\sqrt{\left(\frac{m}{\hat{k}}\right)^2+\frac{9}{4}}, \frac{7}{4}-\frac{1}{2}\sqrt{\left(\frac{m}{\hat{k}}\right)^2+\frac{9}{4}}, \frac{3}{2}, \sin^2\left(\frac{(|z|-z_0)}{\hat{L}}\right)\right) \\ &= F\left(\alpha',\beta',\gamma',\sin^2\left(\frac{(|z|-z_0)}{\hat{L}}\right)\right) \end{aligned} \tag{3.11}$$

C_1, C_2 are constants.

The following expansion for hypergeometric functions proves very useful, especially for small $\cos^2((|z|-z_0)/\hat{L}) = (1-x) = \epsilon$:

$$F(a,b,a+b-m,x) = \frac{\Gamma(m)\Gamma(a+b-m)}{\Gamma(a)\Gamma(b)}(1-x)^{-m}\times$$

$$\times\left[\sum_{k=0}^{m-1}\frac{(a-m)_k(b-m)_k}{k!(1-m)_k}(1-x)^k\right]-(-1)^m\frac{\Gamma(a+b-m)}{\Gamma(a-m)\Gamma(b-m)}\times$$
$$\times\left[\sum_{k=0}^{\infty}\frac{(a)_k(b)_k}{k!(k+m)!}(1-x)^k[\ln(1-x)-\psi(k+1)-\right.$$
$$\left.-\psi(k+m+1)+\psi(a+k)+\psi(b+k)]\right] \tag{3.12}$$

where m is an integer, and $(a)_k = a(a+1)\cdots(a+k-1)$, $(a)_0 = 1$.

3.2.1 Finding the mass

Using the boundary conditions at the branes:

$$\hat{\psi}' - \frac{3}{2\hat{L}}\tan\left(\frac{(|z|-z_0)}{\hat{L}}\right)\hat{\psi} = 0 \tag{3.13}$$

we find the following constraint

$$C_1\left[-4\tan(.)F_{\frac{5}{4}} + 2\sin(.)\cos(.)\frac{\alpha\beta}{\gamma}F_{\frac{9}{4}}\right] =$$
$$= C_2\sin(.)\left[-4\tan(.)F_{\frac{7}{4}} + 2\sin(.)\cos(.)\frac{\alpha'\beta'}{\gamma'}F_{\frac{11}{4}} + \cot(.)F_{\frac{7}{4}}\right] \tag{3.14}$$

where $\sin(.) \equiv \sin((|z|-z_0)/\hat{L})$

Let us relabel $\epsilon = \cos^2((|z|-z_0)/\hat{L})$, and define ϵ_0 as its value at the first brane and ϵ_c as its value at the 2nd brane. ϵ_0 is small, and assuming that ϵ_c is also small (which is certainly true for $z_c \geq 2z_0$) we find upon substituting for the F's using (3.12)that:

$$\left(\frac{|F_{\frac{5}{4}}|}{|F_{\frac{7}{4}}|}\right)\frac{C_1}{C_2} = \left(\frac{\Gamma_{\frac{7}{4}+}\Gamma_{\frac{7}{4}-}}{\Gamma_{\frac{5}{4}+}\Gamma_{\frac{5}{4}-}}\right)\frac{2C_1}{C_2} = -\left(1-\frac{3\epsilon_0}{E}\right) = \left(1-\frac{3\epsilon_c}{E}\right) \qquad \Rightarrow E = \frac{3}{2}(\epsilon_0+\epsilon_c)\,. \tag{3.15}$$

Where we have relabeled $(m\hat{L})^2$ as E and are assuming it is reasonably small ($\ll 1$) for the "ultralight" mode. Thus the mass is given by

$$m = \frac{1}{l_4^2}L\sqrt{\frac{3}{2}\left(1+\frac{\cosh^2\left(\frac{y_0}{L}\right)}{\cosh^2\left(\frac{(y_c-y_0)}{L}\right)}\right)} \sim \frac{1}{l_4^2} \ll \frac{1}{l_4}\,. \tag{3.16}$$

The mass is small compared to the mass of the rest of the KK modes, the smallest of which are of order $m_{KK} \sim 1/l_4$.

We are using the following short-hand notation:

$$|F_{\frac{5}{4}}| \equiv \frac{\Gamma(\frac{1}{2})}{\Gamma_{\frac{5}{4}+}\Gamma_{\frac{5}{4}-}}$$
$$|F_{\frac{7}{4}}| \equiv \frac{\frac{1}{2}\Gamma(\frac{1}{2})}{\Gamma_{\frac{7}{4}+}\Gamma_{\frac{7}{4}-}} \tag{3.17}$$

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where

$$\Gamma_{\frac{7}{4}\pm} = \Gamma\left(\frac{7}{4} \pm \frac{1}{2}\sqrt{E + \frac{9}{4}}\right)$$
$$\Gamma_{\frac{5}{4}\pm} = \Gamma\left(\frac{5}{4} \pm \frac{1}{2}\sqrt{E + \frac{9}{4}}\right). \tag{3.18}$$

Substituting back for E we find the coefficients are related by

$$\left(\frac{|F_{\frac{5}{4}}|}{|F_{\frac{7}{4}}|}\right)\frac{C_1}{C_2} = \frac{\epsilon_0 - \epsilon_c}{\epsilon_0 + \epsilon_c}. \tag{3.19}$$

Where are the wavefunctions concentrated when the 2nd brane is at some general position z_c? Keeping the condition that ϵ_c is small, and expanding the F's we find

$$\begin{aligned}\hat{\psi}^{(1)}(z) = \cos^{-\frac{3}{2}}\Bigg[C_1\left(|F_{\frac{5}{4}}|(1 + \frac{3}{8}\cos^2(.)(\epsilon_0 + \epsilon_c))\right) - \\ - C_2\sin(.)\left(|F_{\frac{7}{4}}|(1 + \frac{\cos^2(.)}{2} + \frac{3}{8}\cos^2(.)(\epsilon_0 + \epsilon_c))\right)\Bigg]\end{aligned} \tag{3.20}$$

so at the branes

$$\hat{\psi}^{(1)}(0) \sim \frac{2\epsilon_0^{\frac{1}{4}}}{\epsilon_0 + \epsilon_c}$$
$$\hat{\psi}^{(1)}(z_c) \sim \frac{-2\epsilon_c^{\frac{1}{4}}}{\epsilon_0 + \epsilon_c}. \tag{3.21}$$

Finally we can also calculate the normalization constants of the wavefunction, provided we assume that E is sufficiently small. Let $C_1 = \alpha C_2$, then:

$$\begin{aligned}\frac{1}{\hat{L}C_2^2} = \Bigg[\alpha^2(|F_{\frac{5}{4}}|)^2\left(\frac{\sin(.)}{\cos^2(.)} + \ln\tan\left(\frac{\pi}{4} + \frac{1}{2\hat{L}}(z - z_0)\right)\right) - \\ - 2\alpha(|F_{\frac{5}{4}}|)(|F_{\frac{7}{4}}|)\frac{1}{\cos^2(.)} + (|F_{\frac{7}{4}}|)^2\left(\frac{\sin(.)}{\cos^2(.)} - \ln\tan\left(\frac{\pi}{4} + \frac{1}{2\hat{L}}(z - z_0)\right)\right)\Bigg]_0^{z_c}.\end{aligned} \tag{3.22}$$

Now let us see what all this means at specific positions of the 2nd brane.

3.2.2 $z_c = z_\infty$

In this case $\epsilon_c \to 0$ and (3.15) tells us that $E = \frac{3}{2}\epsilon_0$. Then

$$E = \left(m\hat{L}\right)^2 = \left(mL\right)^2\cosh^2(\frac{y_0}{L}) \qquad \Rightarrow m = \frac{4}{L}\sqrt{\frac{3}{2}}e^{-\frac{2}{L}y_0} = \frac{1}{l_4^2}L\sqrt{\frac{3}{2}}$$

is the mass of this ultralight mode. This is much smaller than the inverse radius of AdS_4. Also

$$\left(\frac{|F_{\frac{5}{4}}|}{|F_{\frac{7}{4}}|}\right)\frac{C_1}{C_2} = 1$$

so

$$\hat{\psi}^{(1)}(z) = \cos^{\frac{5}{2}} C_2 \left[\left(\frac{|F_{\frac{7}{4}}|}{|F_{\frac{5}{4}}|} \right) F_{\frac{5}{4}} - \sin(.) F_{\frac{7}{4}} \right]$$

now from (3.21) we see that

$$\begin{aligned} \hat{\psi}^{(1)}(0) &\sim 2\epsilon_0^{-\frac{3}{4}} \\ \hat{\psi}^{(1)}(z_c) &\sim \frac{-2\epsilon_c^{\frac{1}{4}}}{\epsilon_0} \sim 0\,. \end{aligned} \tag{3.23}$$

So we find that the ultralight mode is concentrated on the first brane as the 2nd is taken to infinity. Earlier we saw how the massless mode is concentrated on the 2nd brane in this case, and how it is not normalizable. The same is not true of the ultralight mode:

$$\frac{1}{\hat{L}C_2^2} = (|F_{\frac{7}{4}}|)^2 \left(\frac{1}{\cos^2(\frac{\pi}{2})} + \ln\tan(\frac{\pi}{2}) - \frac{2}{\cos^2(\frac{\pi}{2})} + \frac{1}{\cos^2(\frac{\pi}{2})} - \ln\tan\left(\frac{\pi}{2}\right) + \text{finite} \right) \tag{3.24}$$

There is no diveregnce so C_2 is finite. Thus the 1st KK state is normalizable even when the 2nd brane is taken away.

3.2.3 $z_c = 2z_0$

For the symmetric configuration we see that $\epsilon_0 = \epsilon_c = \epsilon$ so $C_1 = 0$ and $\hat{\psi}^{(1)}(z) = C_2 \sin(.) F_{\frac{7}{4}}$ then $E = 3\epsilon \Rightarrow m = \frac{4}{L}\sqrt{3}e^{-\frac{2}{L}y_0} = \frac{1}{l_4^2}L\sqrt{3} = \sqrt{2}m_\infty$.

Also (3.21) tells us the wavefunction is equally concentrated on the two branes for the symmetric configuration.

What happens for a slightly asymmetric configuration?

If z_c is slightly $> 2z_0$ then $\epsilon_c < \epsilon_0 \Rightarrow m < m_{symm}$. For instance, in y coordinates if the second brane is at position $2y_0 + \delta$ then the mass becomes $m = m_0(1 - \delta/2L)$. Also note that $\frac{C_1}{C_2}$ is small and positive. This results in the ultralight mode being concentrated more on the 1st brane.

If z_c is slightly $< 2z_0$ then $\epsilon_c > \epsilon_0 \Rightarrow m > m_{symm}$ and $\frac{C_1}{C_2}$ is small and negative. The mode is more concentrated on the 2nd brane.

So we see the mass of the 1st KK state decreases as the 2nd brane moves further away, all the while coupling more strongly to the 1st brane.

Finally looking at the normalization of the wavefunction for the symmetric configuration, (3.22) shows there is no divergence. Setting $\alpha = 0$ we find

$$\hat{\psi}^{(1)}(0)^2 = \hat{\psi}^{(1)}(2z_0)^2 = \frac{\sin^2(.)\cos^{-3}(.)}{\hat{L}\left(2\frac{\sin(z_0/\hat{L})}{\cos^2((z_0/\hat{L}))} + \ln\left(\frac{\tan(\frac{\pi}{4} + \frac{z_0}{2\hat{L}})}{\tan(\frac{\pi}{4} - \frac{z_0}{2\hat{L}})} \right) \right)} \tag{3.25}$$

We earlier found the wavefunction for the massless mode and so we see that (3.8)

$$\hat{\psi}^{(0)}(0)^2 \approx \frac{1}{2\hat{L}} \frac{1}{\cos(z_0/\hat{L})} \sim \frac{1}{2L} \approx \hat{\psi}^{(1)}(0)^2\,. \tag{3.26}$$

Thus the massless and ultralight gravitons have roughly the same couplings to the branes in this special symmetric configuration.

3.2.4 $z_c = z_0$

Perhaps the most interesting case occurs as the 2nd brane approaches the turn-around in the warpfactor, z_0. In order to find the wavefunction of the 1st KK mode at the 2nd brane, let us use the relation:

$$F(a,b,c,x) = (1-x)^{(c-a-b)} F(c-a, c-b, c, x) \tag{3.27}$$

and then expand F in terms of $x = \sin^2((z_c - z_0)/\hat{L})$ which is very small. We then substitute this into (3.11).

We need to find the boundary condition at z_c and we do this by plugging our value for $\hat{\psi}^{(1)}(z_c)$ into (3.14). We find

$$\begin{aligned} -EC_1 &= C_2 s_c \left(\text{const} - \frac{E}{2} + \frac{1}{s_c^2} + \cdots \right) \qquad \Rightarrow \frac{C_1}{C_2} \sim -\frac{1}{E s_c} \\ s_c &\equiv \sin\left(\frac{(z_c - z_0)}{\hat{L}}\right). \end{aligned} \tag{3.28}$$

If we use (3.28) together with the boundary condition from the brane at 0 (3.15), we find that it is inconsistent to have E small:

$$\frac{C_1}{C_2} = -\frac{1}{E s_c} = \left(\frac{|F_{\frac{7}{4}}|}{|F_{\frac{5}{4}}|}\right)\left(-1 + \frac{3\epsilon_0}{E}\right) \qquad \Rightarrow -\frac{1}{s_c} \sim \text{const}(3\epsilon_0 - E)\,. \tag{3.29}$$

This is impossible as s_c,ϵ_0 and E are all assumed to be very small. Obviously this implies E cannot be exactly zero either.

Is it possible that E is not of order ϵ_0 but is still finite — say $E < 4$ ($E = 4$ for the next KK mode)? We find from the boundary condition at 0 (3.15) that

$$\frac{C_1}{C_2} = -\text{const} \equiv -\frac{1}{s_c E} \tag{3.30}$$

this is impossible as $s_c \to 0$

Say E is very large, or to be more precise

$$E \sim \frac{1}{s_c} \to \infty \tag{3.31}$$

then the boundary condition at $z_c \to z_0$ (3.28) gives us that $C_1/C_2 = -\text{const}$. A lengthy analysis of the boundary condition at 0 in this situation indicates that $C_1/C_2\left(|F_{\frac{5}{4}}|/|F_{\frac{7}{4}}|\right) \sim -1$.

Thus it is consistent to have very large $E \sim 1/s_c$, which blows up as the 2nd brane approaches the turn-around in the warp factor.

We can check that E does indeed increase as we move the brane away from the symmetric configuration at $z_c = 2z_0$ towards z_0. Let us investigate how far the brane must be moved in order to change the order of magnitude of E significantly. For instance, when is $E \sim \epsilon_0^{\frac{1}{2}}$ (that is $m \sim \sqrt{L/l_4^3}$) ? We find this occurs at $\epsilon_c \sim \epsilon_0^{\frac{1}{2}}$, which is equivalent to $z_c = 1.95 z_0$ if we let $y_0/L = 5$. We can also show that $E \sim 1$ (or $m \sim (1/l_4)$) at $z_c = 1.57 z_0$. We find we lose the "ultralight" graviton very quickly.

Where is this 'not so light' 1st KK mode concentrated as $z_c \to z_0$?

For

$$
\begin{aligned}
E &\sim \epsilon_0^{\frac{1}{2}} \\
\hat{\psi}^{(1)}(0) &\sim \left(\frac{\epsilon_0}{\epsilon_c}\right)^{\frac{1}{4}} \epsilon_c^{-\frac{3}{4}} \\
\hat{\psi}^{(1)}(z_c) &\sim \epsilon_c^{-\frac{3}{4}}
\end{aligned}
$$

and for $E \sim 1$

$$
\begin{aligned}
\hat{\psi}^{(1)}(0) &\sim \epsilon_0^{\frac{1}{4}} \\
\hat{\psi}^{(1)}(z_c) &\sim \text{const}\,.
\end{aligned}
$$

So we see the 1st KK state being more and more concentrated at the 2nd brane as z_c decreases and E increases. As $E \to \infty$ we find this effect of $\hat{\psi}^{(1)}(z_c) \gg \hat{\psi}^{(1)}(0)$ becomes even more pronounced. Thus as the second brane approaches the turn-around point in the warp factor, the mass of the previously 'ultralight' mode actually blows up, and this mode is concentrated on the second brane. We have a theory with only a single massless graviton. This is what we expect. As $z_c \to z_0$ we see less and less of the geometry associated with the turn around in the warp factor. As it is the turn around that really differentiates between the physics corresponding to Minkowski and AdS branes, cutting off the space should result in a model similar to RS1.

3.2.5 The other KK modes

As an interesting aside note that not only does the ultralight mode disappear in this limit, but infact we lose exactly half the KK modes when $z_c = z_0$ because of the orbifold symmetry. E is finite so

$$
\begin{aligned}
\frac{C_1}{C_2} &= -\frac{1}{E s_c} \to \infty \qquad \Rightarrow C_2 = 0 \\
\hat{\psi}^{(1)}(z) &= C_1 \cos^{\frac{5}{2}}(.) F_{\frac{5}{4}}\,.
\end{aligned}
\tag{3.32}
$$

This is consistent with the boundary condition at the first brane only if we make $F_{\frac{5}{4}}$ finite, so

$$
E = n(n+3), n \text{ } odd\,. \tag{3.33}
$$

We lose the states with n even — that is those with wavefunctions that are asymmetric about z_0, because of the orbifold condition at that point.

3.2.6 What if $z_c < z_0$?

Let us briefly address the question of what happens when $z_c < z_0$. As earlier stated, we expect to approximately reproduce the results of RS because in this case we cut off the space before reaching the turn-around point in the warp factor. The second brane necessarily has negative tension given by

$$
V_c = \frac{3}{L}\tanh\left(\frac{(y_c - y_0)}{L}\right). \tag{3.34}
$$

Let us suppose $y_c \ll y_0$. We find that we cannot consistently solve the boundary conditions for the first KK state if we assume that it is ultralight:

$$-1 + \frac{3\epsilon_0}{E} + \frac{3\epsilon_0^2}{E} + \cdots = -1 + \frac{3\epsilon_c}{E} + \frac{3\epsilon_c^2}{E} + \cdots \tag{3.35}$$

this is not possible for $\epsilon_0 \neq \epsilon_c$

On the other hand, looking at the zero mode we find that it is still normalizable:

$$\begin{aligned} \hat{\psi}^{(0)}(z) &= \frac{A^2}{\cos((|z| - z_0)/\hat{L})^3} \\ A^2 &= \frac{1}{\hat{L}(\frac{\sin(z_0/\hat{L})}{\cos(z_0/\hat{L})^2} - \frac{\sin((z_0-z_c)/\hat{L})}{\cos((z_0-z_c)/\hat{L})^2})} . \end{aligned} \tag{3.36}$$

As we always have $|z - z_0| < z_0$ we find $\hat{\psi}^{(0)}(z)$ is concentrated on the first brane and falls off towards the second brane. This is just as in the RS scenario. We reiterate that it is the turn-around in the warp factor that leads to the new and exciting results found in the Karch-Randall scenario — if the second brane cuts off the physics before we reach z_0 we won't see an ultralight mode.

4. Implications

4.1 The Newton potential

Now that we have investigated where the two gravtion modes are concentrated, we can finally find the form of the Newton potential between two unit masses on the first brane. We expect that both gravitons will contribute different amounts to the potential depending on the position of the second brane. For instance, because the ultralight mode becomes heavy as $z \to z_0$ we should find that only the massless mode contributes in that case.

If the masses are separated by distance r, then

$$\begin{aligned} V(r) &= \sum_n \frac{1}{r} e^{-mr} |\hat{\psi}^{(n)}(0)|^2 \frac{1}{M^3} \\ &\approx \frac{1}{r}\frac{1}{M^3}(|\hat{\psi}^{(0)}(0)|^2 + e^{-m_1 r}|\hat{\psi}^{(1)}(0)|^2) + \int_2^\infty dn \frac{e^{-m_n r}}{M^3 r} |\hat{\psi}^{(n)}(0)|^2 . \end{aligned} \tag{4.1}$$

For $r \ll 1/m_1 \sim l_4^2/L$ the contribution of the first two gravitons is roughly

$$\begin{aligned} \frac{1}{r}\frac{1}{M^3}|\hat{\psi}^{(0)}(0)|^2 &\equiv \frac{1}{M_{pl0}^2}\frac{1}{r} \\ \frac{1}{r}\frac{1}{M^3}|\hat{\psi}^{(1)}(0)|^2 &\equiv \frac{1}{M_{pl1}^2}\frac{1}{r} . \end{aligned} \tag{4.2}$$

For instance, in the symmetric configuration, $|\hat{\psi}^{(0)}(0)|^2 = \frac{M^3}{M_4^2} \approx |\hat{\psi}^{(1)}(0)|^2$ so that we get $\frac{1}{r}\frac{1}{M_4^2} \equiv \frac{1}{M_{pl0}^2}\frac{1}{r} \equiv \frac{1}{M_{pl1}^2}\frac{1}{r}$.

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Here M_4 is the 4D scale obtained from dimensionally reducing the action. ($M_4^2 = M_5^3 L$) We see that for ultralarge distances $r \gg 1/m_1$ the ultralight mode decouples. Over smaller distances both modes contribute to the Newton potential. For nonsymmetric positionings of the two branes, $\hat{\psi}^{(0)}$ and $\hat{\psi}^{(1)}$ contribute different amounts. For instance, as $y_c \to \infty$ only $\hat{\psi}^{(1)}$ provides a significant contribution ($M_{pl0} \gg M_{pl1}$), as the massless mode disappears, whereas when $y_c \to y_0$, the reverse is true and $\hat{\psi}^{(0)}$ dominates instead.

4.2 The hierarchy problem

One of the original motivations for the RS model was that it solved the hierarchy problem. We can ask whether the same is true of our model. As in Randall-Sundrum, physical masses are rescaled with respect to the fundamental mass parameter m_0 by

$$m = e^{A(y)} m_0 \,. \tag{4.3}$$

If we put the SM fields on the second brane and use some form of stabilization mechanism to fix its position, we will need $y_c < 2y_0$ for $e^{A(y_c)} < 1$ as:

$$e^{A(y)} = \frac{\cosh\left(\frac{|y|-y_0}{L}\right)}{\cosh\left(\frac{y_0}{L}\right)} \,.$$

However we have seen that if $y_c \ll 2y_0$ then only $\hat{\psi}^{(0)}$ mediates 4D gravity and we lose the bigravity characteristic of our model.

4.3 Origins of the 2 gravitons

In the bigravity model we have 2 gravitons combining in such a way that we see effective 4D gravity. The idea that we have 2 gravitons in the theory may seem somewhat mysterious. To better understand the origins of this scenario let us consider the symmetric configuration once more, but with both branes placed at an infinite separation from one another [6].

We then have two independent branes with delta-function potentials, each supporting its own zero mode. When the branes are brought to a finite separation, these two modes mix, and are no longer degenerate in mass.

Earlier we obtained the form of the graviton wave function in the KR model which we will call $\hat{\psi}^{(1)}_\infty$. The graviton is localized on the brane at 0 and the second brane is at ∞. We can obtain the form of the equivalent wavefunction $\hat{\psi}'^{(1)}_\infty$ which is instead centred on a second brane at z_c (while assuming the first brane is at $-\infty$). We have:

$$\hat{\psi}'^{(1)}_\infty(z) \equiv \hat{\psi}^{(1)}_\infty(|z_c - |z||) \,. \tag{4.4}$$

We should find that by taking linear combinations of these two functions, we end up with $\hat{\psi}^{(0)}(z)$ and $\hat{\psi}^{(1)}(z)$:

$$\begin{aligned} \hat{\psi}^{(0)}(z) &= \alpha\hat{\psi}'^{(1)}_\infty + \beta\hat{\psi}^{(1)}_\infty \\ \hat{\psi}^{(1)}(z) &= \gamma\hat{\psi}'^{(1)}_\infty + \delta\hat{\psi}^{(1)}_\infty \,. \end{aligned} \tag{4.5}$$

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Now

$$\hat{\psi}^{(1)}_{\infty} = C_2 \cos^{\frac{5}{2}}\left(\frac{(|z|-z_0)}{\hat{L}}\right)\left[\frac{|F_{\frac{7}{4}}|}{|F_{\frac{5}{4}}|}F_{\frac{5}{4}} - \sin\left(\frac{(|z|-z_0)}{\hat{L}}\right)F_{\frac{7}{4}}\right]$$

and in the symmetric configuration when $z_c = 2z_0$:

$$\begin{aligned}\hat{\psi}'^{(1)}_{\infty}(z) &\equiv \hat{\psi}^{(1)}_{\infty}(|z_c - |z||) \\ &= C_2 \cos^{\frac{5}{2}}((|z|-z_0)/\hat{L})\left[\frac{|F_{\frac{7}{4}}|}{|F_{\frac{5}{4}}|}F_{\frac{5}{4}} + \sin\left(\frac{(|z|-z_0)}{\hat{L}}\right)F_{\frac{7}{4}}\right].\end{aligned} \tag{4.6}$$

Thus we find for the symmetric configuration that

$$\begin{aligned}\frac{\hat{\psi}^{(1)}_{\infty} - \hat{\psi}'^{(1)}_{\infty}}{2} &= -C_2 \cos^{\frac{5}{2}}\left(\frac{(|z|-z_0)}{\hat{L}}\right)\sin\left(\frac{(|z|-z_0)}{\hat{L}}\right)F_{\frac{7}{4}} \\ &\equiv \hat{\psi}^{(1)}(z)\,.\end{aligned} \tag{4.7}$$

What about the orthogonal combination?

$$\begin{aligned}\frac{\hat{\psi}^{(1)}_{\infty} + \hat{\psi}'^{(1)}_{\infty}}{2} &= -C_2 \cos^{\frac{5}{2}}\left(\frac{(|z|-z_0)}{\hat{L}}\right)\frac{|F_{\frac{7}{4}}|}{|F_{\frac{5}{4}}|}F_{\frac{5}{4}} \\ &\equiv \hat{\psi}^{(0)}(z)\,.\end{aligned} \tag{4.8}$$

This is not the form of $\hat{\psi}^{(0)}$ we know and love. However if we impose the boundary conditions on this new wavefunction, we find that it must be massless:

$$\frac{d\psi}{dz} = \left(-\frac{5}{2\hat{L}}\tan(.) + \frac{4}{\hat{L}}\sin(.)\cos(.)\left(1-\frac{E}{4}\right)\frac{F_{\frac{9}{4}}}{F_{\frac{5}{4}}}\right)\psi$$

if $(m\hat{L})^2 = 0$ then $F_{\frac{9}{4}}/F_{\frac{5}{4}} = \cos^{-2}((|z|-z_0)/\hat{L})$ and this satisifies the boundary condition:

$$\frac{d\psi}{dz} = \frac{3}{2\hat{L}}\tan(.)\psi$$

Now once we substitute $E = 0$ the new wavefunction reduces to the familiar form of $\hat{\psi}^{(0)}$:

$$\frac{\hat{\psi}^{(1)}_{\infty} + \hat{\psi}'^{(1)}_{\infty}}{2} \sim \cos^{\frac{-3}{2}}\left(\frac{(|z|-z_0)}{\hat{L}}\right) \equiv \hat{\psi}^{(0)}(z)\,. \tag{4.9}$$

Thus taking linear combinations of the KR ultralight modes located on each brane indeed gives us the two gravitons of the bigravity model. We can see from a different perspective, that the existence of the two gravitons themselves is a result of having two branes on either side of, and far from the turnaround point in the warp factor.

4.4 Minkowski limit

Let us now check the $\Lambda_4 \to 0$ limit for our models. This is equivalent to taking $y_0/L \to \infty$. Is this a valid limit?

We know that M_{pl} is a function of $\hat{\psi}^{(0)}(0)$ and $\hat{\psi}^{(1)}(0)$ (if we're on the first brane), and we want it to be finite. From dimensionally reducing the 5D action we find the 4D scale to be:

$$M_4 = \frac{M^3 L}{\cosh^2(\frac{y_0}{L})}\left(\frac{y_c}{L} + \sinh\left(\frac{y_c}{L}\right)\cosh\left(\frac{y_c - 2y_0}{L}\right)\right).$$

This is easily seen to be finite for $y_c < y_0$, $y_c = y_0$ and $y_c = 2y_0$, even as we send $y_0 \to \infty$.

Now in the KR model, M_4 is infinite, but all the contribution to $\frac{1}{M_{pl}^2}$ comes from $\hat{\psi}^{(1)}(0)^2$, and this is finite, even in the limit $y_0 \to \infty$. Thus this is the correct limit to take.

Let's look at the case $y_c > y_0$ in the limit $y_0 \to \infty$ for $y_c = y_0$, $y_c = 2y_0$ and $y_c \gg 2y_0$. Consider

$$\begin{aligned}\hat{\psi}^{(0)}(z)^2 &= \frac{A^2}{\cos^3\left(\frac{|z|-z_0}{\hat{L}}\right)} \\ &\equiv \frac{\cosh^2(\frac{y_0}{L})}{L(\frac{y_c}{L} + \sinh(\frac{y_c}{L})\cosh(\frac{y_c-2y_0}{L}))}\frac{\cosh^3(\frac{y_c-2y_0}{L})}{\cosh^3(\frac{y_0}{L})}.\end{aligned} \tag{4.10}$$

For finite $|y| < y_0$:

$$\psi^{(0)}(y)^2 \sim \frac{1}{L}e^{\frac{2y_0}{L}}e^{-\frac{y_c}{L}}e^{-\frac{|y_c-2y_0|}{L}}e^{-\frac{3|y|}{L}}. \tag{4.11}$$

Thus for $y_c = 2y_0$, then $\psi^{(0)}(y)^2 \sim \frac{1}{2L}e^{-\frac{3|y|}{L}}$ for $y_c \gg 2y_0$, $\psi^{(0)}(y)^2 \sim \frac{1}{L}e^{-\frac{3|y|}{L}}e^{\frac{4y_0}{L}}e^{-\frac{2y_c}{L}} \to 0$ as before and for $y_c = y_0$, $\psi^{(0)}(y)^2 \sim \frac{1}{L}e^{-\frac{3|y|}{L}}$.

Now let us check the behaviour of $\psi^{(1)}$ in this limit. Its mass goes to zero as $e^{-\frac{2y_0}{L}}$ for $y_c \geq 2y_0$.

For $y_c = y_0$ we found the mass of $\psi^{(1)}$ blows up like $\coth(\frac{y_c-y_0}{L})$ and this does not change as $y_c = y_0 \to \infty$, so this mode is irrelevant.

However for $y_c = 2y_0$,

$$\begin{aligned}\hat{\psi}^{(1)}(z) &= -C_2\cos^{-\frac{3}{2}}\left(\frac{z-z_0}{\hat{L}}\right)|F_{\frac{7}{4}}|\sin(.)\left(1 + \frac{\cos^2(.)}{2} + \cdots\right) \\ \Rightarrow \psi^{(1)}(y) &= -C_2\cosh^{\frac{3}{2}}\left(\frac{y-y_0}{L}\right)|F_{\frac{7}{4}}|\tanh(.)\left(1 + \frac{\cosh^{-2}(.)}{2} + \cdots\right).\end{aligned} \tag{4.12}$$

So $\psi^{(1)}(y)^2 \sim e^{-\frac{3|y|}{L}}$, and it should have the same normalization as $\psi^{(0)}(y)$.

For $y_c \to \infty$ we found that $\psi^{(1)}$, unlike $\psi^{(0)}$, is normalizable, and

$$\begin{aligned}\hat{\psi}^{(1)}(z) &= C_2\cos^{-\frac{3}{2}}\left(\frac{z-z_0}{\hat{L}}\right)|F_{\frac{7}{4}}| \times \\ &\quad \times\left[\left(1 + \frac{\cos^2(.)}{4}E\right) - \sin(.)\left(1 + \frac{\cos^2(.)}{2}(1 + \frac{1}{2}E)\right)\right] \\ \sin\left(\frac{z-z_0}{\hat{L}}\right) &\equiv \tanh(\frac{y-y_0}{L}) \sim -1.\end{aligned} \tag{4.13}$$

So $\hat{\psi}^{(1)}(z) \sim \cosh^{\frac{3}{2}}(\frac{y_0-y}{L}) \sim e^{-\frac{3}{2L}z}$.

Thus we find that we obtain the RS model with $\psi^{(0)}$ and $\psi^{(1)}$ combining to produce the massless RS graviton: for

$$\begin{aligned} y_c = y_0 &\Rightarrow \psi^{(0)} \to \psi^{(0)}_{RS} \\ y_c = 2y_0 &\Rightarrow \psi^{(0)}, \psi^{(1)} \to \psi^{(0)}_{RS} \\ y_c \gg 2y_0 &\Rightarrow \psi^{(1)} \to \psi^{(0)}_{RS} \,. \end{aligned} \tag{4.14}$$

5. Conclusion

We find many interesting conclusions are to be drawn from studying the effects of having 2 gravitons. While for a range of positions z_c of the 2nd brane they both have an effect (with each being concentrated on a different brane, apart from for the symmetric configuration); we see that we will inevitably lose one of these gravitons in certain limits.

When we send the 2nd brane off to infinity, thus ending up with only one brane, we lose the massless mode which then fails to be normalizable. However if we bring the 2nd brane too close to the first — ie when it approaches the turn-around in the warpfactor — the mass of the 1st KK state blows up, and it can no longer fulfil its role as the 2nd graviton.

The interplay of the two gravitons should be open to a holographic interpretation [4] and this provides an interesting possibility for future investigation.

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